BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than four printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Effect of a parametric driving force on noise-induced transitions: Analytical results

J. Plata
Departamento de Física Fundamental y Experimental, Universidad de La Laguna, La Laguna, Tenerife E38204, Spain
(Received 3 August 1998)

The suppression by a parametric harmonic action of noise-induced oscillations in an underdamped pendulum with nonlinear friction, recently reported by Landa *et al.* [Phys. Rev. E **56**, 1465 (1997)], is studied in an approximately soluble model system. In the high-frequency limit, a process of consecutive averaging over two widely different relevant time scales reveals the analogy of the problem with a noise-induced transition whose critical point is changed by the driving term. The obtainment of analytical results for the probability distribution function and the spectrum allows us to understand and control the effect. [S1063-651X(99)04302-0]

PACS number(s): 05.40.−a

The presence of multiplicative noise in nonlinear systems can lead to effects unpredictable from a deterministic approach and specific to the parametric character of the fluctuations. It is remarkable that, in multiplicative stochastic processes, the most probable values of the relevant variables do not necessarily coincide with the deterministic stationary points, and, as a consequence, the threshold conditions for qualitative changes in the probability density can depend not only on the deterministic parameters, as it happens in additive processes, but also on the noise strength. In studies on the emergence of state-dependent fluctuations in the macroscopic dynamics of diverse physical systems, it was shown analytically that, for particular zero-dimensional models, this property leads to the appearance of the analog of an equilibrium phase transition with effective order parameter and critical temperature both depending on the intensity of the fluctuations [1,2]. The phenomenon, which had previously been found by Stratonovich in the study of self-excited oscillations in electronic circuits with ''external noise'' [3], was termed a *noise-induced transition* [4], and its relevance in different contexts has frequently been pointed out [2]. Recently, similar behavior has been found in the oscillations induced by parametric broadband noise in a pendulum [5]; it has also been shown that these oscillations can be suppressed by the action of a parametric harmonic force [6]. The aim of our work has been to understand analytically this suppression. To this end we have focused on an approximately soluble model system that presents all the elements necessary for the occurrence of the effect. The results, which make explicit the connection with a noise-induced transition whose critical point is changed by the driving force, provides us with the clues to control the output signal.

We have considered an underdamped harmonic oscillator parametrically driven by a harmonic action and perturbed by a quartic potential, a nonlinear frictional force, and a parametric broadband noise. Specifically, we have studied the Stratonovich stochastic equation [5,6]

$$\ddot{x}+\epsilon 2\beta(1+\alpha\dot{x}^2)\dot{x}+\omega_0^2[1+a\cos(\omega_a t)+\epsilon^{1/2}\xi(t)](x-\epsilon\gamma x^3)=0, \quad (1)$$

where the presence of the ϵ factor in some of the terms indicates their perturbative character; β and α are parameters of the frictional force; $\xi(t)$ is the wide-band noise; a and ω_a are, respectively, the amplitude and frequency of the driving term; and γ gives the nonlinearity of the potential (for $\gamma=1/6$ the model describes a pendulum with sufficiently small oscillations [5]).

In the high-frequency regime, $\omega_a \gg \omega_0$, there are three widely separate characteristic times in the problem: first, the period of the driving force $\tau_a=2\pi/\omega_a$; second, the period of the unperturbed harmonic oscillator $\tau_0=2\pi/\omega_0$; and third, the time linked to the secular variations of the amplitude and phase of the generated oscillations, t_S, which, because of the ϵ factor in Eq. (1), is much longer than τ_0. In this limit, analytical solutions can be obtained. In effect, we can assume that, given the magnitude of both the noise term and the frictional force, their effect on the dynamics during a driving period is negligible; consequently, the method introduced by Landau in Ref. [7] to study the effect of a fast-forcing term on a Hamiltonian system can be used to average over the driving period the Hamiltonian part of our system. Neglecting second-order terms in ϵ, the thus obtained coarse-grained system corresponds to a harmonic oscillator with an effective frequency given by

$$\omega_{ef}=\omega_0\left(1+\frac{a^2\omega_0^2}{2\omega_a^2}\right)^{1/2} \quad (2)$$

and perturbed by a quartic potential whose effective parameter is

$$\gamma_{ef}=\gamma\left(1+2\frac{a^2\omega_0^2}{\omega_a^2}\right). \quad (3)$$

Hence, the complete reduced system is described by the equation

$$\ddot{x}+\omega_{ef}^2 x=\epsilon[-2\beta(1+\alpha\dot{x}^2)\dot{x}+\omega_0^2\gamma_{ef}x^3]-\epsilon^{1/2}\omega_0^2\xi(t)(x-\epsilon\gamma x^3), \tag{4}$$

which can now be solved following the asymptotic methods developed by Krylov and Bogoliubov in the study of nonlinear oscillations [8] as they were subsequently applied by Stratonovich to stochastic systems [3]. In this sense, we choose as definitions for the amplitude, A, and phase, $\psi=\omega_{ef}t+\varphi$, of the oscillations the equations $x=A\cos(\omega_{ef}t+\varphi)$ and $\dot{x}=-\omega_{ef}A\sin(\omega_{ef}t+\varphi)$.

With these changes, Eq. (4) is reduced to a system of two first-order equations in *standard form* [3], and the average of the deterministic terms over the period of the oscillator $\tau_{ef}=2\pi/\omega_{ef}$ can be readily carried out. Additionally, since $\xi(t)$ is modeled as a zero-mean colored noise centered on the frequency $2\omega_0$, with spectral density

$$S[\xi;\omega]=2\kappa(\omega)=4\lambda\sigma^2\frac{\omega^2+(2\omega_0)^2+\lambda^2}{[\omega^2-(2\omega_0)^2-\lambda^2]^2+4\lambda^2\omega^2} \tag{5}$$

and correlation function

$$k(\tau)=\sigma^2 e^{-\lambda|\tau|}\cos(2\omega_0\tau), \tag{6}$$

it is clear that for a sufficiently small correlation time, $1/\lambda\ll 1/(\epsilon\omega_{ef})$, the average of the stochastic terms can also be performed. Then, after minor algebra we obtain that in first order the averaged equations are [3]

$$\dot{A}=\epsilon\left[\left(-\beta+\frac{\omega_0^4}{8\omega_{ef}^2}\kappa(2\omega_{ef})\right)A-\frac{3}{4}\beta\alpha\omega_{ef}^2A^3+\epsilon^{-1/2}\frac{\omega_0^2}{2\omega_{ef}}A\zeta_1(t)\right],$$

$$\dot{\varphi}=\epsilon\left[-\frac{3\omega_0^2}{8\omega_{ef}}\gamma_{ef}A^2+m+\epsilon^{-1/2}\frac{\omega_0^2}{2\omega_{ef}}\zeta_2(t)\right], \tag{7}$$

where

$$m=-\frac{\omega_0^4\sigma^2}{4\omega_{ef}^2}\left[\frac{\omega_{ef}-\omega_0}{\lambda^2+4(\omega_{ef}-\omega_0)^2}+\frac{\omega_{ef}+\omega_0}{\lambda^2+4(\omega_{ef}+\omega_0)^2}\right]<0, \tag{8}$$

and the effective stochastic forces $\zeta_1(t)$ and $\zeta_2(t)$ are Gaussian white noise terms defined by

$$\langle\zeta_i(t)\rangle=0,$$

$$\langle\zeta_i(t)\zeta_i(t')\rangle=K_i\delta(t-t'),\quad i=1,2, \tag{9}$$

with $K_1=\kappa(2\omega_{ef})/2$ and $K_2=\kappa(0)+\kappa(2\omega_{ef})/2$.

In this framework the time evolution of the amplitude is given by a multiplicative stochastic process typical of the previously mentioned *noise-induced transitions* [1,2]. An important difference with that effect must nevertheless be noted. In our case, the broadband noise $\xi(t)$ affects the amplitude in two ways: through the *deterministic* term $\omega_0^4\kappa(2\omega_{ef})/(8\omega_{ef}^2)A$, which changes the bifurcation point of the deterministic dynamics, and through the stochastic force $\zeta_1(t)$, which, entering multiplicatively in the equation, alters the position of the *deterministic* stationary points. For this process, which has also been shown to be relevant in the study of ‘‘on-off intermittency’’ [9,10], the steady-state probability density can be obtained analytically [1–3,5], and, in terms of the parameters

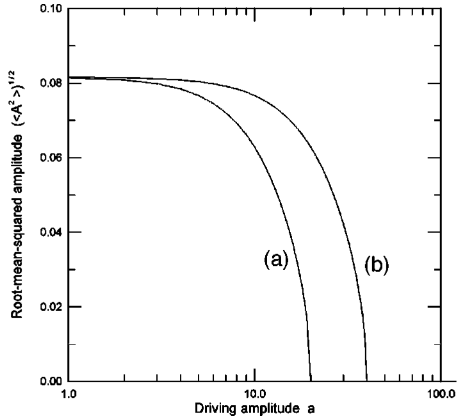

FIG. 1. Root-mean-square amplitude vs the amplitude of the driving force for $\omega_a=20$ (a) and $\omega_a=40$ (b). The rest of the parameters of the system are in both cases $\alpha=100$, $\beta=0.1$, $\lambda=100$, and $\kappa(2\omega_0)/\kappa_{cr}(2\omega_0)=1.5$.

$$\nu=\frac{\dfrac{-8\beta\omega_{ef}^2}{\omega_0^4}+\kappa(2\omega_{ef})}{2K_1}, \tag{10}$$

$$\Lambda=\frac{3\beta\alpha\omega_{ef}^4}{\omega_0^4K_1}, \tag{11}$$

it reads

$$W_{SS}(A)=\begin{cases}\dfrac{2\Lambda^\nu}{\Gamma(\nu)}A^{2\nu-1}e^{-\Lambda A^2} & \text{for } \nu>0\\ \delta(A) & \text{for } \nu\leqslant 0.\end{cases} \tag{12}$$

From these results it is understood how, for certain values of the amplitude and frequency of the parametric harmonic action, the oscillations generated by noise are suppressed. In effect, in order to have oscillations it is necessary that $\nu>0$, which, in the absence of driving, implies having values of the noise intensity larger than the threshold $\kappa_{cr}(2\omega_0)=8\beta/\omega_0^2$. In contrast, when driving is present, the condition for the existence of oscillations is $\kappa(2\omega_{ef})>8\beta\omega_{ef}^2/\omega_0^4$, and, taking into account the functional dependence of ω_{ef} with a and ω_a, it is evident that a higher noise intensity is required to generate the signal. More precisely, the oscillations are suppressed if the amplitude of the driving force exceeds the critical value $a_{cr}^2=2(\omega_a/\omega_0)^2\{\omega_0^2\kappa(2\omega_{ef})/(8\beta)-1\}$. These conclusions, which explain qualitatively part of the findings of Ref. [6], are clearly illustrated in Figs. 1 and 2, where the root-mean-square amplitude of the oscillations $(\langle A^2\rangle)^{1/2}=(\nu/\Lambda)^{1/2}$ is depicted versus the frequency and amplitude of the driving force, respectively. It stands out that it is the quotient between a and ω_a that determines the suppression of the output signal. The fact that $\kappa(2\omega_{ef})<\kappa(2\omega_0)$ contributes also to a rise of the threshold, but, due to the broadband structure of the noise spectrum, only higher-order corrections derive from it. The creation by a high-frequency driving field of *dressed* potentials has previously been used in a different context to implement changes in the dynamics of a stochastic system [11]; in our case the critical point for the onset of the oscillations and the mean

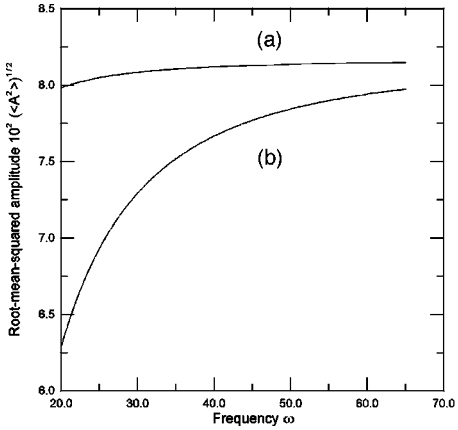

FIG. 2. Root-mean-square amplitude vs the frequency of the driving force for $a=3$ (a) and $a=10$ (b). The rest of the parameters are the same as in Fig. 1.

value of their amplitude can be controlled with a proper choice of the parameters of the driving force.

The time evolution of the phase corresponds to an additive stochastic process. Because of the coupling, due to the nonlinear part of the potential, of A and φ in this process, we cannot obtain analytical solutions for the complete probability density. However, the analysis of the mean frequency

$$\langle \dot{\psi} \rangle = \omega_{ef} + \epsilon \left[-\frac{3\omega_0^2}{8\omega_{ef}} \gamma_{ef} \frac{\nu}{\Lambda} + m \right] \tag{13}$$

gives some information about the characteristics of the generated signal. Contrary to the increase of the frequency caused by the driving term and given by Eq. (2), there is a reduction that has its origin, first, in the nonlinear character of the potential, and second, in the twofold influence of noise: the *deterministic* term m lowers the mean frequency, and the stochastic force $\zeta_2(t)$ changes the peak frequency in the spectrum. This decrease, irrelevant in our model, can be important in a less restrictive regime. Spectral changes with a similar origin have been found in bidimensional stochastic systems with deterministic dynamics inside bifurcation regions [12]. Note that the occurrence in our reduced system of a regime of *fully developed oscillations* [3] can be interpreted as the generation induced by noise of a *limit cycle*.

From this study it is evident that the nonlinearity of the potential is not a necessary condition for the existence of the effect; it is in fact the nonlinear friction with the particular functional form assumed that plays the key role in the appearance of the instability in our model. Therefore, to have insight into the qualitative changes detected in the spectra by varying the noise intensity [5], we set $\gamma=0$ and $\epsilon=1$ in Eq. (7), and, following Refs. [3] and [13], we find an approximation for the spectrum of this new system, which, although not equivalent to our starting model, is suitable to identify the mechanisms responsible for specific spectral features. To this end, we take as correlation time for the amplitude, τ_A, the expression

$$\tau_A = \frac{8\omega_{ef}^2}{\omega_0^4 K_1 \nu} \left(\nu - \frac{\Gamma^2(\nu+1/2)}{\Gamma^2(\nu)} \right), \tag{14}$$

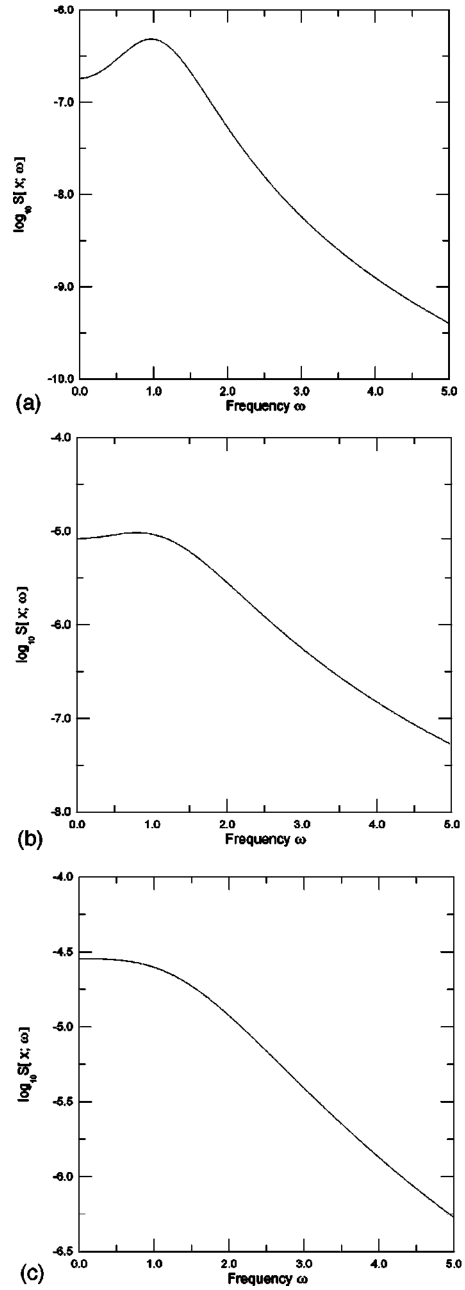

FIG. 3. Approximate spectrum for the reduced system of Eq. (7) with $\alpha=100$, $\beta=0.1$, $\lambda=100$, $\gamma=0$, $\omega_a=20$, $a=3$, and $\kappa(2\omega_0)/\kappa_{cr}(2\omega_0)=1.2$ (a), $\kappa(2\omega_0)/\kappa_{cr}(2\omega_0)=2$ (b), and $\kappa(2\omega_0)/\kappa_{cr}(2\omega_0)=3$ (c).

which was obtained in Ref. [3] applying a decoupling ansatz for the moments, and which, despite its approximate validity [14,15], is useful in a first approach to the problem. In this sense we have used it to obtain the spectrum as

$$S[x;\omega] = \frac{\langle A \rangle^2}{2} \left(\frac{D_1}{(\omega-\omega_{ef}-m)^2 + D_1^2} + \frac{D_1}{(\omega+\omega_{ef}+m)^2 + D_1^2} \right) + \frac{\langle A^2 \rangle - \langle A \rangle^2}{2} \left(\frac{D_2}{(\omega-\omega_{ef}-m)^2 + D_2^2} + \frac{D_2}{(\omega+\omega_{ef}+m)^2 + D_2^2} \right), \tag{15}$$

where $D_1=\omega_0^4 K_2/(8\omega_{ef}^2)$ is the contribution to the width of the signal that comes from the fluctuations in phase and $D_2=D_1+1/\tau_A$ is the total width obtained when fluctuations in amplitude are also considered; obviously, the relative importance of both terms depends on the variance of the amplitude $\langle A^2\rangle-\langle A\rangle^2=[\nu-\Gamma^2(\nu+1/2)/\Gamma^2(\nu)]/\Lambda$.

In Fig. 3 we plot this spectrum for three values of the noise strength. Increasing noise levels give rise to a widening of the signal and to a reduction of the peak frequency. Eventually a qualitative change takes place: coherence is completely lost as the preferred frequency disappears. The fast loss of coherence can be understood as the result of the combined effect of the fluctuations in the two variables. We conjecture that these effects along with the additional shift in frequency due to the nonlinearity of the potential can be relevant in the transitions observed in the spectra of more general models [5].

Concluding, we summarize our results in three main points. First, in the high-frequency limit, the effect of the parametric driving force on the studied system is just a *renormalization* of the potential, the fundamental frequency being larger in the *dressed* potential. As a consequence, the critical point for the instability is changed, higher noise intensities being necessary to have the transition; additionally, the strengths of the effective stochastic forces are reduced as the broadband noise is not centered at the double of the effective fundamental frequency. Second, it is the cooperative effect of the nonlinear friction, with the particular functional form assumed, and the parametric noise that gives rise to the oscillations; the nonlinearity in the potential, which is changed by the driving force, affects in first order the frequency of the output signal, whereas it has only a second-order effect on the position of the critical point. Third, the mean frequency diminishes as the noise strength increases; this property combined with the widening of the signal can partially account for the qualitative changes detected in the spectrum. The generalization of our model by including additive fluctuations is straightforward. Indeed, the analytical results of Ref. [1] reveal that the presence of weak additive noise in the equation for the amplitude has a considerable effect for $\nu<1/2$, leading to the disappearance of the singularity at the origin and therefore preventing the complete suppression of the oscillations [6]; in contrast, it does not produce essential changes in the probability density outside the threshold region. It is worth comparing this behavior with the effect of additive noise on ''on-off intermittency'' [9].

The model presented accounts for the main characteristics of the output signal and gives an analytical criterion for its suppression. The mechanism behind the noise-generated oscillations is the same one responsible for self-excited oscillations in nonlinear systems [3]: the parametric noise alters the deterministic dynamics in an effective way, leading to the onset of a bifurcation at a certain strength, and subsequently changing the amplitude and frequency of the *limit cycle*. The similarity with a *noise-induced transition* [1] is also evident. However, it must be noted that for that phenomenon the *order parameter* of the transition was identified with the most probable value of the relevant variable A_m, which equals $\{(2\nu-1)/(2\Lambda)\}^{1/2}$ for $\nu\geqslant 1/2$ and zero elsewhere [1,3]; in our case the oscillations exist merely when A has nonzero values $(\nu>0)$ [5]. Hence, the thresholds for both processes are defined in a different way. The difference with the so-called *noise-induced nonequilibrium phase transitions*, recently found in the study of spatially distributed systems subject to multiplicative noise [16], is clear: despite the similar terminology, the problems are different.

Finally, we emphasize the necessity of the ϵ factor in the equations to guarantee the difference in time scales and consequently the applicability of the used methodology. In spite of this limitation, the model gives clues to analyze some features of the effect that are still present under less restrictive conditions, and sets up a framework to evaluate the relative importance of the different elements that can be incorporated in the modeling of real physical problems. Since state-dependent noise emerges in a natural way in the description of diverse processes, or it can conversely be included in the dynamics in an externally controllable way, the possible relevance of the phenomenon in a wide variety of contexts is clear.

[1] A. Schenzle and H. Brand, Phys. Rev. A **20**, 1628 (1979).
[2] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).
[3] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).
[4] For a discussion about the validity of this terminology, see H. Brand, A. Schenzle, and G. Schröder, Phys. Rev. A **25**, 2324 (1982).
[5] P. S. Landa and A. A. Zaikin, Phys. Rev. E **54**, 3535 (1996).
[6] P. S. Landa, A. A. Zaikin, M. G. Rosenblum, and J. Kurths, Phys. Rev. E **56**, 1465 (1997).
[7] L. D. Landau and M. Lifshitz, *Mechanics* (Pergamon Press, Oxford, 1960).
[8] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations* (Gordon and Breach, New York, 1961).
[9] N. Platt, S. M. Hammel, and J. F. Heagy, Phys. Rev. Lett. **72**, 3498 (1994).
[10] H. L. Yang and E. J. Ding, Phys. Rev. E **54**, 1361 (1996).
[11] J. Plata, Phys. Rev. E **57**, 5154 (1998).
[12] J. Plata, Phys. Rev. E **56**, 6516 (1997).
[13] J. D. Cresser, W. H. Louisell, P. Meystre, W. Schleich, and M. O. Scully, Phys. Rev. A **25**, 2214 (1982).
[14] J. M. Sancho, M. San Miguel, S. L. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589 (1982).
[15] R. Graham and A. Schenzle, Phys. Rev. A **25**, 1731 (1982).
[16] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).